UNIT - II

Section 2.4 Divergent Sequences

Defn 2.4 A: ~~Div~~ Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that s_n approches infinity as n approaches infinity, if for any real number $M>0$ there is a positive integer N such that

$$s_n \geq M \quad (\forall\ n \geq N)$$

we can write $s_n \to \infty$ as $n \to \infty$.

Ex sequence $\{n\}_{n=1}^{\infty}$ diverges to ∞.

For given $M>0$, just choose $N \in I$ such that $N \geq M$. Then certainly $n \geq M$ $\forall\ n \geq N$

Definition 2.4B: Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers, we say that s_n approches to minus infinity as 'n' approches to infinity if for any real number $M>0$ there is a positive integer N such that

$$s_n < -M \quad (\forall\ n \geq N)$$

then we write $s_n \to -\infty$ as $n \to \infty$.

or $\{s_n\}_{n=1}^{\infty}$ diverges to minus infinity.

Ex: The sequence $\left\{\log\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$ diverges to minus infinity.

Let us ~~that~~ choose

$N > e^M$

$\therefore \forall\ n \geq N \Rightarrow n \geq N > e^M$

$\log_e\left(\frac{1}{n}\right) < -M$

$\frac{1}{n} < e^{-M}$

$n > e^M$

$\Rightarrow n > e^{M}$

$\Rightarrow \log\left(\frac{1}{n}\right) < -M \quad \forall n \geq N$

$\therefore$ The sequence $\left\{\log\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$ diverges to minus infinity.

2.5 Bounded Sequence.

Definition 2.5A: $\{s_n\}_{n=1}^{\infty}$ is bounded if and only if there exists $M \in R$ such that $|s_n| \leq M \quad (\forall n \in I)$

Theorem 2.5B. If the sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ is convergent, then $\{s_n\}_{n=1}^{\infty}$ is bounded.

(or)

Prove that every convergent sequence is bounded sequence.

Proof Given the sequence $\{s_n\}_{n=1}^{\infty}$ is convergent.

Say $\{s_n\}_{n=1}^{\infty}$ converges to L.

$\Rightarrow \lim\limits_{n\to\infty} s_n = L$

By the defn given $\varepsilon > 0$ there exists $N \in I$ such that

$$|s_n - L| < \varepsilon \quad (\forall n \geq N)$$

$$|s_n - L| < 1 \quad (\forall n \geq N) \quad \because \varepsilon < 1$$

For $|s_n| = |s_n - L + L|$
$\leq |s_n - L| + |L|$
$< 1 + |L|$

~~$-1 < s_n - L < 1$~~

$\Rightarrow \quad |s_n| < L + 1 \quad \forall n \geq N. \quad \text{——— } (1)$

Let us assume $M = \max\{|s_1|, |s_2| \cdots |s_{N-1}|\}$

$[s_1, s_2, s_3 \cdots s_{N-1}, s_N, s_{N+1}, \cdots$

$\Rightarrow |s_n| < M+L+1 \quad \forall n \in I$

Let $M+L+1 = K$ $\quad$ ~~$K > 0$~~ $K \in R$.

$|s_n| < K \quad \forall n \in I$

$\Rightarrow$ sequence $\{s_n\}_{n=1}^{\infty}$ is bounded sequence.

2.6 Monotone Sequence.

Defn Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers

(i) If $s_1 \leq s_2 \leq s_3 \leq \ldots \leq s_n \leq s_{n+1} \leq \ldots$, then $\{s_n\}_{n=1}^{\infty}$ is called nondecreasing. (or) (increasing seq) sequence.

(ii) If $s_1 < s_2 < s_3 < \ldots < s_n < s_{n+1} < \ldots$ then seq $\{s_n\}_{n=1}^{\infty}$ is strictly increasing sequence.

(iii) If $s_1 \geq s_2 \geq s_3 \geq \ldots s_n \geq s_{n+1} \geq \ldots$ then the sequence $\{s_n\}_{n=1}^{\infty}$ non-increasing sequence (or) decreasing sequence.

(iv) If $s_1 > s_2 > s_3 > \ldots s_n > s_{n+1} > \ldots$ then the sequence $\{s_n\}_{n=1}^{\infty}$ is strictly decreasing Sequence.

2.6 B **Theorem** A nondecreasing Sequence which is bounded above is convergent. (4)

(or)

Prove that increasing sequence which is bounded above is convergent.

Proof:

T.P $\lim_{n\to\infty} s_n = M$

(or) T.P if given $\varepsilon > 0$ $\exists$ $N \in I$ such that

$|s_n - M| < \varepsilon \quad \forall n \geq N$

$-\varepsilon < s_n - M < \varepsilon \quad \forall n \geq N$

$M - \varepsilon < s_n < M + \varepsilon \quad \forall n \geq N$

Given $\{s_n\}_{n=1}^{\infty}$ is increasin sequence

$s_1 \leq s_2 \leq \ldots s_N \leq s_{N+1} \leq \ldots$ ——— (1)

also given $\{s_n\}_{n=1}^{\infty}$ is bounded above.

$\{s_n\}_{n=1}^{\infty} = \{s_1, s_2, s_3 \ldots\}$. this set is bounded above set.

[If a set is bounded above then it has a l.u.b in R]

$\therefore$ the set $\{s_1, s_2, s_3 \ldots\}$ has a l.u.b

Let M be the l.u.b of the set $\{s_1, s_2, s_3 \ldots\}$

$\Rightarrow \quad s_n \leq M \quad \forall n \in I$

$\Rightarrow \quad s_n < M + \varepsilon \quad \forall n \in I$ ——— (1)

Since M is the l.u.b of the set $\{s_1, s_2, s_3 \ldots\}$

$\therefore$ $M - \varepsilon$ is not an upper bound.

$\Rightarrow$ there exists for some $N \in I$ such that $s_N > M - \varepsilon$

$\therefore$ using (1)

$$s_n > M - \varepsilon \qquad \forall\, n \geq N \quad \text{—— (2)}$$

using (1) & (2)

$$\therefore \quad M - \varepsilon < s_n < M + \varepsilon \qquad \forall\, n \geq N.$$

$$\Rightarrow \quad -\varepsilon < s_n - M < \varepsilon \qquad \forall\, n \geq N$$

$$\Rightarrow \quad |s_n - M| < \varepsilon \qquad \forall\, n \geq N$$

$$\Rightarrow \quad \lim_{n \to \infty} s_n = M$$

$\Rightarrow$ sequence $\{s_n\}_{n=1}^{\infty}$ is converges to M.

$\Rightarrow$ Sequence $\{s_n\}_{n=1}^{\infty}$ is convergent.

2.6 C: Prove that the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$ is convergent.

Here

$$s_n = \left(1+\frac{1}{n}\right)^n \qquad (1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \cdots$$

$$s_n = \left(1 + \frac{n}{1!}\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots \infty\right)$$

$n = 1$

$$s_1 = 1 + 1 = 2.$$

$n = 2$

$$s_2 = 1 + \frac{2}{1!}\frac{1}{2} + \frac{\cancel{2}(1)}{\cancel{2!}}\left(\frac{1}{2}\right)^2 = 1 + 1 + \frac{1}{4} = 2.25$$

$n = 3$

$$s_3 = 1 + \frac{3}{1!}\frac{1}{3} + \frac{3(2)}{2!}\left(\frac{1}{3}\right)^2 + \frac{3(2)\cdot 1}{3!}\left(\frac{1}{3}\right)^3 =$$